AP Calculus BC 6.5 Logistic Growth
Objective: able to solve problems involving exponential or logistic population growth.

**Partial Fraction Decomposition with Distinct Linear Denominators**
If \( f(x) = \frac{P(x)}{Q(x)} \) where \( P \) and \( Q \) are polynomials with the degree of \( P \) less than the degree of \( Q \), and if \( Q(x) \) can be written as a product of distinct linear factors, then \( f(x) \) can be written as a sum of rational functions with distinct linear denominators.

1. Write \( f(x) = \frac{2x+16}{x^2+x-6} \) as a partial fraction decomposition.

**Antidifferentiating with Partial Fractions**
2. \( \int \frac{x^2-6}{x^2-9} \, dx \) (notice the degree of the numerator is not less than the degree of the denominator)
The Logistic Differential Equation
Now consider the case of a population $P$ with a growth curve as a function of time. We have seen that the exponential growth at the beginning can be modeled by the differential equation $\frac{dP}{dt} = kP$ for some $k > 0$ (and equivalently $y = y_0 e^{kt}$). However, not all populations continue to grow without bound. The population begins increasing and concave up (as in exponential growth), then turns increasing and concave down as it approaches the **carrying capacity**, $M$, of its habitat. A logistic curve, like the one shown in the figure, has the shape to model this growth.

If we want the growth rate to approach zero as $P$ approaches a maximal **carrying capacity**, $M$, we can introduce a limiting factor of $M - P$.

The logistic differential equation is: $\frac{dP}{dt} = kP(M - P)$

Consider a (positive) population $P$ that satisfies $\frac{dP}{dt} = kP(M - P)$, where $k$ and $M$ are positive constants.

A. For what values of $P$ will the growth rate $\frac{dP}{dt}$ be close to zero?

B. As a function of $P$, $y = kP(M - P)$ has a graph that is an upside-down parabola. What is the value of $P$ at the vertex of that parabola?

C. Use the answer to part (B) to explain why the growth rate is maximized when the population reaches half the carrying capacity.

D. If the initial population is less than $M$, is the initial growth rate positive or negative?

E. If the initial population is greater than $M$, is the initial growth rate positive or negative?

F. If the initial population equals $M$, what is the initial growth rate?

G. What is $\lim_{t \to \infty} P(t)$? Does it depend on the initial population?
3. The growth rate of a population $P$ of bears in a newly established wildlife preserve is modeled by the differential equation $\frac{dP}{dt} = 0.0002P(1200 - P)$, where $t$ is measured in years.
   
   a. What is the carrying capacity for bears in this wildlife preserve?
   
   b. What is the bear population when the population is growing the fastest?
   
   c. What is the rate of change of the population when it is growing the fastest?

The General Logistic Formula

The solution of the general logistic differential equation

$$\frac{dP}{dt} = kP(M - P)$$

is

$$P = \frac{M}{1 + Ae^{-(Mk)t}}$$  

(See Homework Problem #35 for proof)

where $A$ is a constant determined by an appropriate initial condition. The carrying capacity $M$ and the growth constant $k$ are positive constants.

4. The number of students infected by measles in a certain school is given by the formula $P = \frac{200}{1 + e^{5.3 - t}}$ where $t$ is the number of days after students are first exposed to an infected student.

   a) Show that the function is a solution of a logistic differential equation. Identify $k$ and the carrying capacity.
   
   b) Estimate $P(0)$. Explain its meaning in the context of the problem.
5. A 2000-gallon tank can support no more than 150 guppies. Six guppies are introduced into the tank. Assume that the rate of growth of the population is \( \frac{dP}{dt} = 0.0015P(150 - P) \) where time \( t \) is in weeks.
   a) Find a formula for the guppy population in terms of \( t \).
   b) How long will it take for the guppy population to be 100?

Rate yourself on how well you understood this lesson.

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<th>I sort of get it</th>
<th>I understand most of it but I need more practice</th>
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What do you still need to work on?